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On Causal Dynamics Without Metrisation: Part III

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Abstract

A formalisation of heuristic and intuitive ideas about causal dynamical transitions, presented in Part II of this series (this volume, pp. 1–22), is given here. A quotient structure is found to be necessary for any collection of causal dynamical transitions by using arguments employing the Principle of Corporate Agreement. The relevance of the first homotopy group (the fundamental group) to 'small scale' phenomena predicted by a theory is pointed out, and the fundamental group is seen to play a wide role in defining the causality, relational and quotient structures of a theory.

1. *Introduction*

1.1. *First Considerations*

Is there *too much* information in a theory of dynamics that is dependent upon the set of all ordered pairs of events that are apprehended by observers? One may fairly ask that question of the approach to dynamics that was proposed in Part II of this series. The purpose of this paper is to show that considerable simplification of the information problem can be achieved through the application of the Principle of Corporate Agreement, also proposed in Part II.

The simplification is not merely a 'first order approximation' but the introduction of equivalence relations, thereby allowing details of information about 'equivalent' events or 'equivalent' dynamical transitions to be retained without encumbering the simplified structure.

Moreover, the simplification is not trivial. The notion of 'language equivalence' introduced permits one to convert diagrams (such as Fig. 1 representing the set of all events detected by all observers together with the associated set of ordering operators between the events) into their 'universal covering diagrams'. This is done by using the notion of a base space and its universal covering space. This also means that no simpler diagrams—and therefore no simpler ordering relations—may be constructed. Furthermore it means that if an ordering relation formalism is used for dynamics, then it can be no simpler, nor yet more informative, than this approach. In other words the methods used here may well be more correctly described as *the* (universal) ordering relation formalism for causal dynamics.

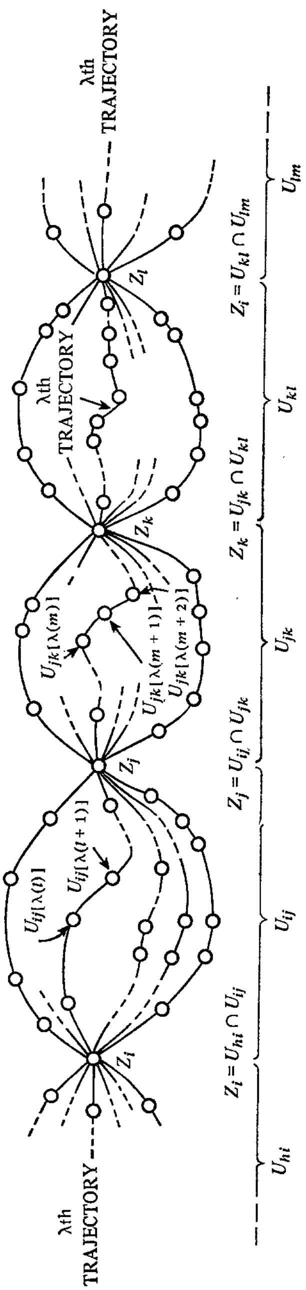
λth TRAJECTORY
λth TRAJECTORY
λth TRAJECTORY
Z_i
Z_j
Z_k
Z_l
$U_{ij[\lambda(t)]}$
$U_{ij[\lambda(t+1)]}$
$U_{jk[\lambda(m)]}$
$U_{jk[\lambda(m+1)]}$
$U_{jk[\lambda(m+2)]}$
$Z_i = U_{hi} \cap U_{ij}$
$Z_j = U_{ij} \cap U_{jk}$
$Z_k = U_{jk} \cap U_{kl}$
$Z_l = U_{kl} \cap U_{lm}$
U_{hi}
U_{ij}
U_{jk}
U_{kl}
U_{lm}

Figure 1.

1.2. *Outline*

The first section of the paper is devoted to giving a set theoretic description of the simple ideas (represented in Fig. 1) concerning sets of events apprehended and ordered by different observers. The notion of an ordering operator associated with an ordering relation of a set is more clearly defined than in Part II. Language equivalence between sets of ordered observed events is defined, and polyadic ordering relations are discussed as a part of understanding the construction of an ordering operator from an ordering relation.

The second section is devoted to interpreting the rôle of the notion of 'fundamental group' (in the sense of the first homotopy group) in a system of causal dynamical transitions. It is found that the fundamental group has a very considerable part to play in defining the quotient structure that is required by the use of the Principle of Corporate Agreement and the notion of language equivalence. The fundamental group—via its property of factoring out a base space from its universal covering space—is also seen to define the dynamical causal ordering relation and dynamical causal ordering operator; also it defines the structure of the transitions of the unobserved 'internal/microscopic' events described by a theory. The fundamental group of the system of dynamical transitions is therefore central to all similar consideration of dynamics. Finally, an example of a quotient theory language is sketched.

2. *Language Equivalence*

2.1. *Introductory Remarks*

In order to consider how the notion of binary ordering relation may be extended so as to be usefully employed in considering the non-metrical formulation of dynamical theories, we need to combine two points of view. The first is the purely formal one developed in this section, and the second is the interpretative form of the method† principally dealt with in Section 3. This section deals with formalisation of the notion of deformation of ordering operators into one another, which was first introduced in Section 4.1, Part II, et seq.

2.2. *Ordering Relations and Ordering Operators*

For the purposes of this discussion let us make the following hypotheses:

HYP(1): There exists a set of observers $\mathfrak{O} = \{\mathcal{O}_\lambda\}$, $\lambda \in I \subset J_\infty^{(+)}$.

HYP(2): Each observer $\mathcal{O}_\lambda$ records and relates his observations by means of a particular set of symbols and notions, called a (λ-) theory-language $£_\lambda$, *the subscript denoting the correspondence between an observer and a theory-language.*

HYP(3): The aggregate of events observed by the λth observer, $\mathcal{O}_\lambda$, constitutes a set, denoted by U_λ.

† Using the notion of intermediate events and processes.

NTN(1): The elements of U_λ are denoted as $u_{\lambda(l)}$; i.e.

$$U_\lambda = \{u_{\lambda(l)} : l \in I_\lambda \subset J_\infty^{(+)}\} \equiv \{u_{\lambda(l)}\}.$$

HYP(4): The aggregate of all events apprehended by the set of all observers, $\mathfrak{O} = \{\mathcal{O}_\lambda\}$, constitutes a set U, such that each U_λ is a proper subset of U and such that $U = \cup\, U_\lambda$.

Let us now appeal to common experience (compare Section 3.1, Part II), and the well-ordering theorem, and suppose that:

HYP(5): The λth set U_λ may be put into a total order with a first element min U_λ and a last element max U_λ; the ordering will be denoted by R_λ. The pair (U_λ, R_λ) is called the λth (individual) trajectory *of the set U.*

RMK(1): At this point it must be remarked, as in Section 3.5, Part II, that the hypothesis of the well-ordering of individual trajectories, which has just been made in HYP(5), does not immediately coincide with the everyday experience of each one of us. This is because if each one of us were to write a record of our remembered experiences—even of experiences, like thoughts, that may have only just occurred—we cannot invariably recall the precise order in which some of them actually occurred. We can, however, generally say that the set of '*ordering confused*' events lies '*after*' one event and '*before*' a '*later*' event. We shall present enough ideas here to show that for present purposes we need not worry about such awkward subsets, the U_λ, for it is intended to say that a general theory-language £, which can relate all the consciously well-ordered subsets of the $U_\lambda \subset U$, must be compared against the 'ordering confused' subsets of the U_λ. If then it is found after repeated experiments by all the $\mathcal{O}_\lambda$ that the common theory-language £ cannot explain a residue of the 'ordering confused' subsets, then it will be apparent that £ must be given a richer structure. Since [compare Section 3, and Section 3.6, DF(3) in Part II] we define a *causal* dynamical theory to be a one that can well order at least a subset of $U = \cup_\lambda U_\lambda$ in a manner agreeable to all the $\mathcal{O}_\lambda$, it is clear that if the ordering used by this theory cannot embrace all the elements of U, then a more detailed, or completely new, ordering relation in U must be found, upon which all $\mathcal{O}_\lambda$ can agree.

CVN(1): If the elements $u_{\lambda(l)} \in U_\lambda$ are ordered into a total order by R_λ, with the integers $l \in I_\lambda$ belonging to a natural sequence such that, if $u_{\lambda(l_i)} = \min U_\lambda$ and $u_{\lambda(l_f)} = \max U_\lambda$, then $l_i \leqslant l \leqslant l_f$, $\forall\, l \in I_\lambda$, we shall denote the ordering of the $u_{\lambda(l)}$ as follows:

$$u_{\lambda(l)} R_\lambda u_{\lambda(m)} \Leftrightarrow l < m$$

HYP/NTN(6): To express the notion of a dynamical transition *from event $u_{\lambda(l)}$ to event $u_{\lambda(m)}$, we denote the* dynamical ordering operator *for the λth trajectory as $\check{R}_\lambda$, and write:*

$$u_{\lambda(m)} = \check{R}_\lambda(u_{\lambda(l)}) \Leftrightarrow l < m$$

An inverse operator is defined by means of the expression:

$$u_{\lambda(l)} = \check{R}_\lambda^{-1}(u_{\lambda(m)}) \Leftrightarrow l < m$$

This condensed notation for a dynamical ordering operator, and its inverse, on a set of events, results from the following simple considerations. Consider a pair of adjacent events, that is to say each is either the immediate predecessor or immediate successor of the other. For example, in a more elaborate notation than so far used we may write:

$$u_{\lambda(l)} R_{\lambda(l,l+1)} u_{\lambda(l+1)} \Rightarrow \begin{cases} u_{\lambda(l+1)} = \check{R}_{\lambda(l,l+1)}(u_{\lambda(l)}) \\ u_{\lambda(l)} = \check{R}_{\lambda}^{-1}{}_{(l,l+1)}(u_{\lambda(l+1)}) \end{cases}$$

Now consider the following simple extension of this notation:

$$u_{\lambda(l+p)} = \check{R}_{\lambda(l+p-1,l+p)} \ldots \check{R}_{\lambda(l,l+1)}(u_{\lambda(l)}) = \check{R}_{\lambda(l,l+p)}(u_{\lambda(l)})$$
$$u_{\lambda(l-q)} = \check{R}_{\lambda}^{-1}{}_{(l-q,l-q+1)} \ldots \check{R}_{\lambda}^{-1}{}_{(l-1,l)}(u_{\lambda(l)}) = \check{R}_{\lambda}^{-1}{}_{(l-q,l)}(u_{\lambda(l)})$$
$$u_{\lambda(l+p-q)} = \check{R}_{\lambda}^{-1}{}_{(l+p-q,l+p)} \check{R}_{\lambda(l,l+p)}(u_{\lambda(l)}) = \check{R}_{\lambda(l,l+p-q)}(u_{\lambda(l)})$$

Consequently all the dynamical ordering operators between the elements of U_λ form a group if there is also defined an *identity operation* in the fashion:

$$\check{R}_{\lambda(E)} \equiv \check{R}_{\lambda(l,l)} = \check{R}_{\lambda}^{-1}{}_{(l,l)} \qquad \text{if} \qquad l \in I_\lambda$$

and if there is also defined an *annihilation operation* according to:

$$\check{R}_{\lambda(A)} \equiv R_{\lambda(l,r)} = \check{R}_{\lambda}^{-1}{}_{(r,m)} \check{R}_{\lambda(l,m)}, \qquad \forall l, m \in I_\lambda, \qquad r \notin I_\lambda$$

so that
$$\check{R}_{\lambda(A)}(U_\lambda) = \phi$$

The group so constructed will be called the *ordering operator group*.

On the basis of this argument, we suppose:

HYP(7): The dynamical ordering operators $\check{R}_\lambda$, $\check{R}_\lambda^{-1}$ [*compare HYP(6)*], *are interpreted as appropriate elements of the* ordering operator group, *constructed above, which is denoted as* $\check{\mathbf{R}}_\lambda$.†

RMK(2): We have now been able to associate a reasonable mathematical meaning to the λth trajectories (U_λ, R_λ)—or writing them in a form more relevant to dynamics, $(U_\lambda, \check{R}_\lambda)$. It is not difficult to show that when $\check{\mathbf{R}}_\lambda$ is applied to U, it generates an equivalence class which is none other than the λth trajectory U_λ.

NTN(2): Denote the set of individual trajectories in U by $\mathscr{U} = \{U_\lambda : \lambda \in I \subset J_\infty^{(+)}\}$; i.e. $U_\lambda \in \mathscr{U}$.

RMK/DF(3): Consequently we may consider that $\check{\mathrm{R}}_\lambda$, which operating on U, has an *associated projection operator* $\mathscr{R}_\lambda$, which projects according to the rule; $\mathscr{R}_\lambda(\mathscr{U}) = U_\lambda$: this enables us to write $U = \bigcup_\lambda \mathscr{R}_\lambda(\mathscr{U})$.

2.3. *Language Equivalence*

Let us return to the theory-language formalisms $£_\lambda$ associated with the observers $\mathscr{O}_\lambda \in \mathfrak{O}$ and individual trajectories $U_\lambda \in \mathscr{U}$.

† For example one can write a typical equation $\check{R}_\lambda(u_{\lambda(l)}) = u_{\lambda(m)} = \check{R}_\lambda^{-1}(u_{\lambda(n)})$, $l \leqslant m \leqslant n$; $l, m, n \in I_\lambda$.

HYP(8): To each theory-language $\pounds_\lambda$ *there is associated a mapping* $\check{\pounds}_\lambda: U_\lambda \to \pounds_\lambda$ *and* $\check{\pounds}_\lambda:\check{\mathrm{R}}_\lambda \to \pounds_\lambda$; *and we write:*

$$\check{\pounds}_\lambda(u_{\lambda(l)}) \subset \pounds_\lambda; \qquad \check{\pounds}_\lambda(\check{\mathbf{R}}_\lambda) \equiv \check{\mathbf{R}}_\lambda^{(\pounds\lambda)} \subset \pounds_\lambda$$

and $\pounds_\lambda$ *has a law of composition* $\overset{\lambda}{\square}$ *such that* $\pounds_\lambda \overset{\lambda}{\square} \pounds_\lambda \subset \pounds_\lambda$.

The Principle of Corporate Agreement [compare Section 3.6, ASSN(10) in Part II] requires that, if the set of observers $\mathfrak{O} = \{\mathcal{O}_\lambda\}$ is to be able to create a study anything like physics, amongst the collection $\mathcal{U} = \{U_\lambda\}$ of individual trajectories there must be some events upon which they can agree not only in interpretation, but that there also exists a formal theory-language which they all interpret in precisely the same way. That is to say:

HYP(9): There exists a subset $Z \subset U$ *such that for every* $z_i \in Z = \{z_i : i \in \mathscr{I} \subset J_\infty^{(+)}\}$, *then* $z_i \in U_\lambda$ *for all* $\lambda \in I$, *and that there exists a theory-language* $\pounds$ *with associated map* $\check{\pounds}$, *together with* $[I]$ *mappings* $\check{\pounds}_\lambda : \pounds_\lambda \to \pounds$, *where* $[I]$ *is the cardinality of I, such that there holds:*

$$\check{\pounds}(z_i) = \check{\pounds}_\lambda \check{\pounds}_\lambda(z_i), \qquad \forall i \in \mathscr{I}, \qquad \lambda \in I$$

Although all the subsets $U_\lambda \subset U$ are totally ordered, it does not mean to say that Z be causally ordered, [compare Section 3.6, DF(3), in part II] it is necessary to select a subset from Z according to the construction of Section 2.2, DF(2), TH(2), in Part II. For convenience we shall denote this causally ordered subset as Z also. The supposition of HYP(9) coupled with this redefinition of Z leads one to suppose that there may be a set of events upon which less than the whole set of observers may agree, and we may express such agreement by the following notion of *language equivalence*:

DF(1): Given at least two individual trajectories $(U_\lambda, \check{\mathbf{R}}_\lambda)$, $(U_\mu, \check{\mathbf{R}}_\mu)$, with respective theory-languages $\pounds_\lambda$, $\pounds_\mu$ and respective associated mappings $\check{\pounds}_\lambda$, $\check{\pounds}_\lambda$, $\check{\pounds}_\mu$, $\check{\pounds}_\mu$, such that there hold:

$$\check{\pounds}_\lambda \check{\pounds}_\lambda(U_\lambda) \subset \pounds_{\Delta(U)} \supset \check{\pounds}_\mu \check{\pounds}_\mu(U_\mu)$$
$$\check{\pounds}_\lambda \check{\pounds}_\lambda(\check{\mathbf{R}}_\lambda) \subset \pounds_{\Delta(\mathrm{R})} \supset \check{\pounds}_\mu \check{\pounds}_\mu(\check{\mathbf{R}}_\mu)$$

where $\pounds_{\Delta(U)}$, $\pounds_{\Delta(\check{\mathbf{R}})}$ are 'simply connected' subsets of $\pounds$, then the dynamical trajectories are said to be *£-equivalent*.

This terminology is clearly justifiable, because when the relations above hold, one may infer that there exist dynamical ordering operators between the elements of U_λ and U_μ, and between the elements of $\check{\mathbf{R}}_\lambda$ and $\check{\mathbf{R}}_\mu$, such that $(U_\lambda, \check{\mathbf{R}}_\lambda)$ may be smoothly transformed into $(U_\mu, \check{\mathbf{R}}_\mu)$. From Section 4.6, TH(3 and 4) in Part II we may draw the example of G_0-equivalence for two time-ordered sequences of events, thus showing that the notion of £-equivalence is non-vacuous. As a result we may partition $\mathcal{U}$ into £-equivalence classes of dynamical trajectories.

NTN/DF(3): The set of *£-equivalence classes* of the set of trajectories $\mathcal{U}$

is denoted by $\mathscr{U}$, and $\mathfrak{U}$ is generated by the £-*equivalence relation* $\mathfrak{R}$, whence we have:

$$\mathfrak{U} = \mathscr{U}/\mathfrak{R}; \qquad \mathfrak{R} = \mathscr{U}/\mathfrak{U}$$

RMK/DF(4): Denote the £-*equivalence class* of $\mathscr{U}$, typified by U_λ, as $\mathscr{U}_\lambda \in \mathfrak{U}$; then one has,

$$\mathscr{U}_\lambda = \{\mathscr{R}_\lambda(U) = U_\lambda : \text{£}_\lambda \check{\text{£}}_\lambda(U_\lambda) \subset \text{£}_{\Delta(U)}\}$$

which in turn allows one to define another 'projection operator' $\mathfrak{R}_{(\lambda)}$ by the condition;

$$\mathfrak{R}_{(\lambda)} \equiv \oplus_{\mu \in (\lambda)} \mathscr{R}_\mu : \mathscr{U} \to \mathscr{U}_\lambda \in \mathfrak{U}$$
$$\mathscr{U}_\lambda = \mathfrak{R}_{(\lambda)}(\mathscr{U}) = \bigcup_{\mu \in (\lambda)} \{U_\mu\}$$

where (λ) is the subset of $\mu \in I$ which label the $U_\mu \in \mathscr{U}$ which are £-equivalent to U_λ, also with $\lambda \in (\lambda)$, and where $\{U_\mu\}$ is the singleton whose element is the set $U_\mu \in \mathscr{U}$.

2.4. *Polyadic Orderings*

We may now review our approach to this problem and consider those events, only, amongst all those belonging to U, that can be encompassed by the scheme of relationships so far defined as being consented to by all observers in $\mathfrak{O} = \{\mathscr{O}_\lambda\}$—that is to say we restrict ourselves to conformity with Section 3.6, ASSN(10), DF(3), Part II, the Principle of Corporate Agreement and the definition of a causal ordering. From this point we may proceed by two stages, the first is to assign a meaningful polyadic ordering relation between all elements of U, which is done in this subsection, and the second stage is to deduce a quotient structure for the dynamical ordering between causally related physical events,† which is done in Section 3, following.

According to the construction of Section 2.2, DF(2), Part II, the set $Z \subset U$ separates the set U into subsets U_{ij} with the properties:

$$\sup U_{ij} = z_j = \inf U_{jk}, \qquad z_j \in Z, \qquad i<j<k, \qquad i,j,k \in \mathscr{I} \subset I \subset J_\infty^{(+)}$$
$$\min Z = \inf U_{ab} = z_a, \qquad \max Z = \sup U_{cd} = z_d,$$
$$a<b \leqslant i<j<k \leqslant c<d, \qquad a,b,c,d \in \mathscr{I}$$
$$(U_{ij} - \{z_j\}) < \{z_j\} < (U_{jk} - \{z_k\})$$

where $A < B$ means that the set A strictly precedes B in the causal ordering. These properties say that the sets U_{ij} consist of all possible intermediate events—in the sense of the causal ordering—between the events z_i and z_j such that $z_i < z_j$. Whilst the z_i, z_j are considered to be the events which are actually observed by physicists $\mathscr{O}_\lambda$ and are related in a satisfactory way by a common theory-language £, there is the possibility that a series of

† 'Physical' in the sense that they result from application of the Principle of Corporate Agreement to the set U.

refined experiments (designed to detect finer observational detail) could reveal a causal sequence of intermediate events, between z_i and z_j that could not be related at all in terms of £.† Therefore, if we wish to use the dynamical causal ordering relation/operator in the set Z, consisting of commonly experienced and interpreted events, to predict new observations on a more detailed scale, it must be able to account for all the possible distinct routes from z_i to z_j consistent with the structure of £; that is to require that such causal dynamical ordering relations/operators to be constructed from the R_λ and $\check{\mathbf{R}}_\lambda$ which refer to the set $U_{ij} \subset U$.

RMK/NTN(5): Denote the groups of ordering operators acting on the trajectories in the various U_{ij} by $\check{\mathbf{R}}_\lambda(ij)$. Then it is clear that over U_{ij}, the direct product,

$$\check{\mathbf{R}}(ij) \equiv \otimes_\lambda \check{\mathbf{R}}_\lambda(ij)$$

is also a group of ordering operators, albeit in a somewhat crude sense, owing to the many products of elements which are equal to the annihilation element. It is also apparent that, in terms of all the sets $U_{ij} \subset U$, there can be defined an ordering operator group which is the direct product of all the $\check{\mathbf{R}}(ij)$, namely:

$$\check{\mathbf{R}} \equiv \otimes_{(ij)} \check{\mathbf{R}}(ij)$$

This group, then, contains all the operators that can take one element of U into another element by operations which follow trajectories.

RMK/NTN(6): Now each z_i is related to another by a group of operators, each element of which is an '*extremal*' element of an $\check{\mathbf{R}}(ij)$; extremal in the sense that it connects $\inf U_{ij}$ with $\sup U_{ij}$. Denote this group of extremal operators as $\check{\mathscr{R}}(ij)$. This denotation, based on the preceding notations and discussions, makes it easy to see that if one *suppose* that a set U_{ij} describe the maximum amount of dynamical phenomena the theory-language £ is capable of inferring as a possible dynamical process leading from event/observation z_i to event/observation z_j, then the £-equivalence classes of U_{ij}, typically written $\mathscr{U}_{\lambda(ij)} \subset \mathscr{U}_\lambda$, have associated 'extremal' products of ordering operators that act as the generators for the group of ordering operators which operate upon $Z \subset U$; denote these 'extremal' products for £-equivalence classes of U_{ij} by $\mathfrak{R}_{(\lambda)}(ij)$. [That these elements be described as generators of a larger group is clearly implied by the hypothesis that $(U_{ij}, \check{\mathbf{R}}(ij))$ contain the maximum possible amount of dynamical information describable by £: one can then also freely assume that $z_i \rightarrow z_j$ is the smallest observable (i.e. detectable) physical change‡ in an event z_i—because $\check{\mathbf{R}}(ij)$ does not explicitly refer to any particular set of numbers other than those that can be elicited from a measuring apparatus—which assumption requires that the theory-language £ then

† This possibility, in the case where £ is adequate, gives rise to the second viewpoint mentioned in Section 2.1; namely the introduction of intermediate, virtual dynamical processes and states.

‡ i.e. a change that all observers agree upon.

describe large-scale changes by repeated application of the smallest possible change; for otherwise £ would be inconsistent.†] The whole group of such ordering operators may be denoted as $\breve{\mathfrak{R}}$, as one would expect owing to the partitioning of $\mathscr{U}$ into £-equivalence classes $\mathscr{U}_\lambda \in \mathfrak{U}$ by the £-equivalence relation $\mathfrak{R}$. We can in fact say that $\mathfrak{R}$ induces $\breve{\mathfrak{R}}$, because we have assumed—as a result of the Principle of Corporate Agreement—that all the trajectories converge together whenever they pass through the elements of $Z \subset U$. This convergence allows us to consider *cross-over* between £-inequivalent trajectories, at an observation (i.e. event in our terminology here), as being representable by products of elements of $\breve{\mathfrak{R}}$: the cross-over operation clearly shows that the £-equivalence structure of $\breve{\mathfrak{R}}$ is very rich in most possibilities. The analogy with projection operators may also be given in order to be complete, viz.;

$$\mathfrak{R}_{(\lambda)} \equiv \oplus_{\mu \in (\lambda)} \mathscr{R}_\mu \Rightarrow \breve{\mathfrak{R}}_{(\lambda)}(ij) \equiv \otimes_{\mu \in (\lambda)} \breve{\mathfrak{R}}_\mu(ij) \Rightarrow \breve{\mathfrak{R}}_{(\lambda)} \equiv \otimes_{(ij)} \breve{\mathfrak{R}}_{(\lambda)}(ij)$$

Therefore the stated aim of this subsection has been accomplished, namely the association with a set of elements identified as physical events, of a polyadic ordering relation which may be interpreted as a group of dynamical transitions between the events.

RMK(7): This approach has also shown that one may expect definite classes of transformations representing inequivalent dynamical processes. Furthermore, the foregoing discussion has specified no more stringent property of a theory-language £ other than that it have certain 'simply-connected' regions. The results are, therefore, with this exception, completely general.

3. *The Fundamental Group*

3.1. *Introductory Remarks*

Here we pursue the notion of £-equivalence in combination with the Principle of Corporate Agreement in order to produce some notions of structure in the ordering operator $\breve{\mathbf{R}}$ that make $\breve{\mathfrak{R}}$, the £-equivalence ordering operator associated with $\mathfrak{R}$, appear like the fundamental group of a group of ordering operators having $\breve{\mathbf{R}}$ as its universal covering space. All the necessary notions are defined as they are needed. This approach results from a consideration of a diagram like that of Fig. 1.

3.2. *Non-extremal Products*

So far it has been shown that to the set of events Z, common to all observers $\mathscr{O}_\lambda \in \mathfrak{O}$ and put into the same causal order by each observer $\mathscr{O}_\lambda$, there can be adduced a group of ordering operators between the elements of Z; furthermore, that the form of this group is governed by the groups of ordering operators for individual trajectories which 'converge' upon and 'diverge' from each $z_i \in Z$. On the basis of the Principle of Corporate

† This argument is reminiscent of the quantum mechanical idea of a transition between two states being contributed to by all possible transition processes.

Agreement the set Z is called the *set of physical events* in U, and in a natural way the theory-language £ is called the *physical theory*. But it is *not* correct to call the ordering operators of the group $\check{\mathbf{R}}$ the 'physical' ordering operators, because $\check{\mathbf{R}}$ includes all the ordering relation structure between the 'non-physical' events in U–Z. The *physical* ordering operators are those which produce transitions between events upon which all observers agree *regardless of the intermediate process* (i.e. a route from one physical event to another physical event). This requirement is expressed and met by considering a *quotient structure* in $\check{\mathbf{R}}$. To make this explicit consider the following definition using notations already defined:

DF(2): Given an ordered set of events $(U, \check{\mathbf{R}})$ with trajectories $U_\lambda \in \mathscr{U}$ and a causally ordered set of events $Z = \bigcap_\lambda U_\lambda$, then a *non-extremal product* of ordering operators is typically a product of the ordering operators between elements of $(\bigcup_\lambda U_{ij[\lambda]}) - \{z_i\} \cup \{z_j\}$ (where $U_{ij[\lambda]}$ is the subset of U such that $z_i = \inf U_{ij[\lambda]}$ for all λ, and $z_j = \sup U_{ij[\lambda]}$ for all λ), together with products of ordering operators which may take a trajectory up to one of z_i, z_j, but not both.

HYP(10): The physical ordering operators *are the elements of the group of all ordering operators modulo the group of non-extremal products. The group of physical ordering operators will be denoted as* $\tilde{\check{\mathbf{R}}}$, *the tilde indicating the quotient structure.*

RMK(8): The elements of $\tilde{\check{\mathbf{R}}}$ may be interpreted as transitions between physical, observed events without any intermediate events being involved in the transition, and yet the transition is still dependent upon the transitions between intermediate events that are allowable in terms of the theory-language £. This latter property is manifested by the existence of £-equivalence classes of products of elements of $\check{\mathbf{R}}$. (An example drawn from the recent work of Haag and Kastler will be sketched in Section 3.6.)

NTN/RMK(4): One can easily see that the non-extremal products of ordering operators (for the individual trajectories) between two adjacent elements of Z—being adjacent in the sense of the causal ordering which Z defines—form proper subgroups, e.g. of $\check{\mathbf{R}}_{ij[\lambda]}$. Let us denote such depleted forms of $\check{\mathbf{R}}_{ij[\lambda]}$ by $'\check{\mathbf{R}}'_{ij[\lambda]}$ each apostrophe signifying the necessary omissions at the 'front' and 'rear' of non-extremal products. We may consequently write:

$$'\check{\mathbf{R}}'_{ij} = \oplus_\lambda {}'\check{\mathbf{R}}'_{ij[\lambda]}$$

Consequently, the physically intersecting group of ordering operators is,

$$\tilde{\check{\mathbf{R}}}_{ij} = \check{\mathbf{R}}_{ij}/'\check{\mathbf{R}}'_{ij} \tag{3.2.1}$$

RMK(9): There arises, here, the difficulty that $'\check{\mathbf{R}}'_{ij}$ is not a normal subgroup of $\check{\mathbf{R}}_{ij}$ because of the way in which we have defined the groups of ordering operators associated with the trajectories of observers. As a result, we cannot properly consider $\tilde{\check{\mathbf{R}}}_{ij}$, as it is defined above in equation

(3.2.1), as a quotient group.† We need some adjustment of argument to overcome this difficulty: a difficulty introduced more by the naïvety of the approach than by any inherent weakness. We proceed by making the following hypothesis which is a variant, or derivative of HYP(10);

HYP((10)): For any particular theory of causal dynamical processes, any transformation which is represented by non-extremal products of ordering operators is unobservable, in the sense that it produces no physical‡ transformation between events.

Let us denote by $\check{\mathbf{R}}_{(Z)}$ the set of transformations between the elements of the set Z of physical events, and let us denote all the other transformations as $\check{\mathbf{R}}_{(U-Z)}$: the notation is obvious. The ineffectiveness of $\check{\mathbf{R}}_{(U-Z)}$ upon $\check{\mathbf{R}}_{(Z)}$ is expressed by the relations:

$$\left.\begin{aligned}(\check{\mathbf{R}}_{(U-Z)} \circ \check{R}^{(2)}_{(Z)}) \circ (\check{\mathbf{R}}_{(U-Z)} \circ \check{R}^{(1)}_{(Z)}) &= \check{\mathbf{R}}_{(U-Z)} \circ (\check{R}^{(2)}_{(Z)} \circ \check{R}^{(1)}_{(Z)}) \\ (\check{R}^{(2)}_{(Z)} \circ \check{\mathrm{R}}_{(U-Z)}) \circ (\check{R}^{(1)}_{(Z)} \circ \check{\mathrm{R}}_{(U-Z)}) &= (\check{R}^{(2)}_{(Z)} \circ \check{R}^{(1)}_{(Z)}) \circ \check{\mathrm{R}}_{(U-Z)}\end{aligned}\right\} \begin{array}{l}\text{For } \check{R}^{(i)}_{(Z)} \in \check{\mathbf{R}}_{(Z)}, \\ \quad i = 1, 2\end{array}$$

These equations state that provided the product $\check{\mathrm{R}}^{(2)}_{(Z)} \circ R^{(1)}_{(Z)}$ is defined, then any unobservable intermediate transition between the transitions $R^{(1)}_{(Z)}$ and $R^{(2)}_{(Z)}$ can be accounted for by means of another unobservable transition either before or after the compounded effect of $R^{(1)}_{(Z)}$ and $R^{(2)}_{(Z)}$.¶

In this way, it is seen that $\check{\mathbf{R}}_{ij}$ is required to be considered as a normal subgroup of $\check{\mathbf{R}}_{ij}$, and hence allowing $\check{\mathbf{R}}_{ij}$ to exist between any pair of elements $z_i, z_j \in Z$. By extension of the argument, we can associate a group '$\check{\mathbf{R}}$', with the whole set U, that is normal. Consequently for the set U of all observed events, having the group $\check{\mathbf{R}}$ of ordering operators, we have:

HYP/DF(10): The group of causal physical ordering operators *is defined as* $\check{\check{\mathbf{R}}} \equiv \check{\mathbf{R}}/'\check{\mathbf{R}}'$.

3.3. *Physical Theory Languages*

Now let us consider these matters in terms of the common theory-language £. Any element of $\check{\mathbf{R}}_{(Z)}$ which is either left- or right-compounded with an element of $\check{\mathbf{R}}_{(U-Z)}$ becomes an element of $\check{\mathbf{R}}_{(U-Z)}$. That is to say in general terms we may write:

$$\check{£}(\check{\mathbf{R}}_{(Z)} \circ \check{\mathbf{R}}_{(U-Z)}) \subset \check{£}(\check{\mathbf{R}}_{(U-Z)})$$
$$\check{£}(\check{\mathbf{R}}_{(U-Z)} \circ \check{\mathbf{R}}_{(Z)}) \subset \check{£}(\check{\mathbf{R}}_{(U-Z)})$$

RMK/ASSN(10): If we assume—as is natural to do, if we want to use

† This is in the usual (and normal) sense of multiplying cosets after the style $(HA_i)(HA_j) = H(A_iA_j)$ where $H \subset G$ is a normal subgroup of G and $A_i, A_j \in G$ but $A_i, A_j \notin H$. In fact, in this case the 'coset multiplication' composition is not faithful.

‡ Physical in the sense of satisfying the Principle of Corporate Agreement.

¶ This is, of course, an assumption that is made explicitly to be able to consider $\check{R}_{(U-Z)}$ as a normal subgroup. It may on occasions be invalid. But see Section 3.4.

elements of £ in correspondence with elements of $\check{\mathbf{R}}$†—that we can perform a factorisation process in £ as follows:

$$\check{£}(\check{\mathbf{R}}_{(Z)} \circ \check{\mathbf{R}}_{(U-Z)}) = \check{£}(\check{\mathbf{R}}_{(Z)}) \,\square\, \check{£}(\check{\mathbf{R}}_{(U-Z)}) \subset \check{£}(\check{\mathbf{R}}_{(U-Z)})$$
$$\check{£}(\check{\mathbf{R}}_{(U-Z)} \circ \check{\mathbf{R}}_{(Z)}) = \check{£}(\check{\mathbf{R}}_{(U-Z)}) \,\square\, \check{£}(\check{\mathbf{R}}_{(Z)}) \subset \check{£}(\check{\mathbf{R}}_{(U-Z)})$$

where $\square$ is a law of composition in £ as required, then it is easy to see that $\check{£}(\check{\mathrm{R}}_{(U-Z)})$ is a two-sided ideal in £.

NTN/DF(5): The two-sided ideal $£(\mathbf{R}_{(U-Z)})$ will be denoted as $\$ \equiv \check{£}(\check{\mathbf{R}}_{(U-Z)})$.

RMK/HPY(11): In view of Section 3.2 preceding, it is justifiable to identify $\check{£}(\text{'}\check{\mathbf{R}}\text{'}) = \$$.

Consequently, the theory-language which is of physical interest is $\check{£}(\check{\check{\mathbf{R}}})$. If we make the natural assumption that £ has a part which is faithfully realised by $\check{\mathbf{R}}$, then we may make the following assumption:

ASSN(11): A CAUSAL physical theory-language *is given by* $\mathfrak{G} \equiv \check{£}(\check{\check{\mathbf{R}}}) \equiv \check{£}(\check{\mathbf{R}})/\check{£}(\text{'}\check{\mathbf{R}}\text{'})$, *or more simply we may adopt the notation following as equivalent:*

$$\mathfrak{C} \equiv £/\$‡$$

RMK/NTN(12): Associated with the theory-language $\mathfrak{C}$ is a canonical mapping $\mathfrak{G}: £ \to £/\$$ which is caused by an equivalence relation, in £, that will also be denoted by $\mathfrak{C}$. Furthermore, notice that this equivalence relation is obtained from $\mathfrak{C} = \check{£}(\mathfrak{R})$. The canonical mapping $\mathfrak{C}$ gives rise to other canonical mappings $\check{\mathfrak{C}}_{(U)}: \check{\mathbf{R}} \to \check{\check{\mathbf{R}}}$, and $\mathfrak{C}_{(U)}: U \to Z$, using here convenient (and suggestive) notations.

3.4. *Fundamental Groups*

It is now desirable to show that the notion of 'fundamental group'—in the sense of the group of homotopy classes of loops in a space—may be directly drawn out of the approach we have used, consisting of associating dynamical transformations (processes) with ordering operators. This facet of our approach comes directly out of applying the notion of £-equivalence to trajectories in U.

To begin with we may depict the relationships between some of the notations which have been associated with the set of all events U by a diagram of the kind in Fig. 1. In particular let us now consider a typical subset $U_{ij} \subset U$ which possesses some intersecting individual trajectories; for example, see Fig. 2. We consider the situation in which—for argument's sake—three individual trajectories leaving z_i all pass through a common

† This assumption of a factorisation in £ by the law of composition $\square$ in £ is natural in that it allows expression of the notion that a sequence of dynamical transformations/processes may be written as a sequence of elements of £ compounded with one another.

‡ If one wishes to extend the monetary analogy, one may call any theory-language $\mathfrak{C}$ defined in this way a 'golden' or 'standard' theory language. Without trying to be nationalistic in any way, one may even call it, humorously, '*the gold standard*'.

event u_P before they all pass through z_j. In this situation it is clear that in terms of the theory languages $£_{\lambda,\mu,\nu}$ we can write:

$$\check{£}_\lambda \check{£}_\lambda(u_P) = \check{£}_\mu \check{£}_\mu(u_P) = \check{£}_\nu \check{£}_\nu(u_P)$$

Let us further assume that these three portions of the individual trajectories are £-equivalent, so that they define an equivalence class of trajectories from z_i to u_P. Next, adjoin (in turn and separately) to this £-equivalence class of trajectories the £-equivalence classes of trajectories in U_{ij}, which lead from u_P to its successors, e.g. $u_{Q(\lambda)}$, $u_{Q(\mu)}$, $u_{Q(\nu)}$. Then take the space of all £-equivalence classes of trajectories in U_{ij} which start from z_i and give it a topology in which neighbourhoods are defined by the £-equivalence classes $(z_i, u_P) * (u_P, u_Q)$, where the bracket denotes the trajectory together

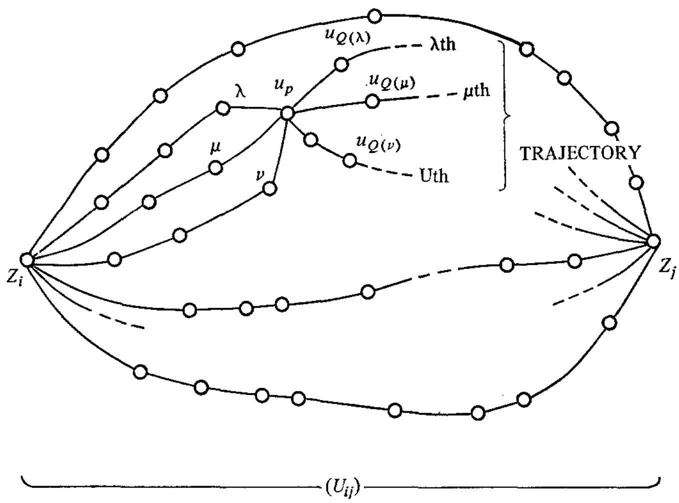

Figure 2.

with its end points, and the $*$ denotes composition of trajectories in the manner of joining paths together. By studying the definition DF(3) given below, it will be verified that we have constructed *the* universal covering space of U_{ij}; hence by extension the universal covering space of U is also constructed. In what immediately follows the actual notation used by Pontryagin (1939) is adjusted to highlight the significance of the symbols (and notions they represent) which have been introduced in these discussions.

The space of events U is not simply connected owing to the presence of £-equivalence classes, but it is easy to see that it is both locally connected and locally simply connected if one takes a neighbourhood of an event to consist of some immediate predecessors and successors together with the transition maps between the events. The local pre-ordering of U gives the necessary validation of deformation properties.

DF(3): Let U be a connected, locally connected, locally simply connected space, and let z be one of its points. Let $\mathscr{U}$ be the set of all paths of U which begin at z, and divide the set $\mathscr{U}$ into homotopy classes, denoting the set so obtained by $\mathfrak{U}$. [There does in fact exist a natural mapping $\hat{\mathfrak{R}}$ of this set onto the space U: for if $\mathscr{U}_\lambda \in \mathfrak{U}$, then all the paths which belong to the class $\mathscr{U}_\lambda$ end in the same point u_λ, and one writes $u_\lambda = \mathfrak{R}(\hat{\mathscr{U}}_\lambda)$]. A topology is now introduced into $\mathfrak{U}$ by defining an arbitrary neighbourhood $\mathscr{V}$ of the topological space $\mathfrak{U}$ in terms of a certain neighbourhood V of the space U, and a certain path $\lambda \in \mathscr{U}$ which ends in V. Let χ be an arbitrary path in V whose initial point coincides with the end of the path λ. Let $\xi = \lambda * \chi$, and let Ξ be the totality of all the paths homotopic to the path ξ. Denote by $\mathscr{V}$ the set of all classes Ξ obtained from all possible choices of χ in V. (Notice that $\mathscr{V}$ does not change if λ is replaced by $\lambda' \in \mathscr{U}_\lambda$, where $\mathscr{U}_\lambda \in \mathscr{V}$.) The totality of all neighbourhoods of the type $\mathscr{V}$ obtained by an arbitrary choice of a neighbourhood V and a path λ forms by definition a complete system Υ of neighbourhoods of the space $\mathfrak{U}$. The space $\mathfrak{U}$ with its topology Υ is called *the universal covering space of U.* [Pontrjagin, Section 46, DF(43).]

We may also note the following property of $\hat{\mathfrak{R}}$, this map being very similar to the dynamical ordering operator $\hat{\mathfrak{R}}: U \to U$;

TH(1): The natural map $\hat{\mathfrak{R}}: \mathfrak{U} \to U$ is a continuous open mapping, i.e. it is interior; moreover it is locally homeomorphic.] Pontrjagin (1939), Section 46, TH(58).

RMK(13): This theorem means that, locally, neighbourhoods of events in U correspond with (i.e. have the same topological properties as) the £-equivalence classes of dynamical transformations from other events into the closer neighbourhoods. And this further shows that in a local sense we can safely study the properties of dynamical transformations in order to examine the properties of physical conditions. Notice that by Section 3.2, TH(*8*), published in Part I of this series,† if $\hat{\mathfrak{R}}$ is one to one, then it is not only a local homeomorphism, but a homeomorphism. We may also note the following important property of the universal covering space:

TH(2): The universal covering space $\mathfrak{U}$ of a topological space U is always simply connected.] Pontrjagin (1939), Section 46, TH(59).

RMK/DF(14): A topological group G has the *universal covering group* $\mathfrak{G}$, which is obtained by constructing the universal covering space $\mathfrak{G}^*$ of the topological space G, taking the identity e of the group G for the fundamental point z of DF(3). Also there exists a natural map $\mathfrak{R}: \mathfrak{C} \to G$ which is a continuous open mapping. The group multiplication operation is introduced into $\mathfrak{G}^*$ in this manner: let A, B be any two elements of the set $\mathfrak{G}^*$, λ, μ,

† *International Journal of Theoretical Physics*, Vol. 1, No. 2, p. 133.

paths in the classes A, B, respectively, both paths beginning at $e \in G$, and their end being denoted a, b, respectively. Then the multiplication is defined by:

$$\mathfrak{R}(A) = a; \qquad \mathfrak{R}(B) = b; \qquad \mathfrak{R}(AB) = \mathfrak{R}(A)\mathfrak{R}(B)$$

This procedure does, in fact, make $\mathfrak{G}^*$ into a topological group owing the following theorem:

TH(3): For every topological group G there exists a simply connected topological group $\mathfrak{G}$ which is locally isomorphic to G, and is such that G is isomorphic to the factor group $\mathfrak{G}/\Pi$, where Π is a discrete normal subgroup of $\mathfrak{G}$, and the fundamental group of G, $\pi(G)$, is isomorphic with the group Π.] Pontrjagin (1939), Section 47, TH(61).

RMK(15): We have therefore shown that it is possible to consider the group of physical, causal, dynamical operators $\breve{\mathbf{R}}$ as having a fundamental group which is determined by the structure of the products of all possible intermediate processes between observed events. However, because it has not been proved that '$\breve{\mathbf{R}}$' is discrete, it cannot be said that our analysis so far is compelling of acceptance; the inference is merely that '$\breve{\mathbf{R}}$' is discrete if one accepts as plausible the notions which have been used throughout, namely that dynamical processes from one physical condition to another may be construed as paths the totality of which does always have a fundamental group. Notice that the fundamental group of a space lists all the possible equivalent ways of travelling from one point to another. Therefore one is forced, eventually, to consider the fundamental group of the space of transitions between physical conditions. The assertion of this essay is that the notions presented provide the most simple and natural way of understanding and formulating this very problem of analysis.

It is possible, before proceeding to further analysis of the nature of the fundamental group in this analysis, to make a comment, RMK(16), which acts as a warning.

TH(4): Let $\mathfrak{C}$ be a simply connected topological group. If a connected topological group $\mathfrak{C}_1$ is locally isomorphic to $\mathfrak{C}$, $\mathfrak{C}_1$ is isomorphic to the factor group of $\mathfrak{C}$ by a discrete subgroup of the centre of $\mathfrak{C}$.] Chevalley (1946), Chapter II, Section VII Scholium.

RMK(16): If experiments (i.e. observations) indicate that a set of physical dynamical processes is invariant under a connected group of transformations, then one may not immediately assume that there are not more complicated invariances which might be detected. For observations are of a local character, hence the invariance observed may not be an absolute invariance, but in fact a 'quotient invariance'. For example, instead of considering invariance under the Poincaré group to be global property of quantum field theory,† it is possible to consider that it is in fact a quotient

† It is of course immediately obvious that the Poincaré group cannot be considered as describing any general invariance, because it has a proper subgroup—the Lorentz group—which is not simply connected. General invariance must be in the form of the existence of a universal covering group, which is always simply connected.

invariance given by the factor group E/S, where E is the invariance group of the formalism and S is the internal symmetry group [compare Michel (1964)].

RMK(17): We may further notice a very fundamental property of the £-equivalence approach adopted here. The universal covering space is that covering space of a given space which cannot be covered by any other space. In other words every simply connected covering space of a given space is a homeomorph of *the* universal covering space of the given space. Therefore $\mathfrak{U}$ really does represent the simplest covering space of the space of trajectories, provided that the description of dynamics in terms of a $(U, \check{\mathbf{R}})$ formalism is the simplest possible way of describing dynamics. It is asserted that this is actually the case, in view of the formulation of $(U,\mathbf{R})$ in terms of categories presently being undertaken.

It is now necessary to take note of the following theorem with respect to the normality of the subgroup of a universal covering space isomorphic to the fundamental group of the covered space.

TH(5): Let G be a locally connected, locally simply connected and simply connected group, and let $H \subset G$ be a discrete subgroup not necessarily normal. Then the fundamental group of the space G/H is isomorphic to H.] Pontrjagin, Section 47, EX(61).

TH(6): Every arcwise-connected topological group has an abelian fundamental group.] Hocking & Young (1960), COR(4.19).

The first theorem admits the possibility that a topological group may have a non-abelian fundamental group; the second one tells us that such topological groups may not be arcwise-connected, i.e. that at least one class of elements of the group cannot be expressed as a continued product of infinitesimal elements starting from other elements of the group.

RMK(18): Since the Poincaré group is not arcwise-connected it is immediately asked whether its fundamental group is abelian or non-abelian.

3.5. *Fundamental Groups as Operators*

We have seen how a fundamental group may be ascribed to the set of observed events (one can also posit 'observable' events), and the reader may have noticed how the fundamental group—as portrayed by '$\check{\mathbf{R}}$'—appears as an operator between events. Here we note the role of the fundamental group as an operator in the universal covering space of a topological space, following Hilton and Wylie (1960).

DF(4): If $p:\mathfrak{U} \to U$ is a covering map of the space U with universal covering space $\mathfrak{U}$ [compare DF(3), TH(1)], then a *cover transformation* is an autohomeomorphism $h:\mathfrak{U} \to \mathfrak{U}$ such that $ph = p$.

RMK(19): The cover transformations form a group H of operators on $\mathfrak{U}$. The space may be regarded as the space $\mathfrak{U}/H$ generated by $\mathfrak{U}$ and the equivalence relation that associates $\mathfrak{u}$ and $h(\mathfrak{u})$ for each $\mathfrak{u} \in \mathfrak{U}$ and $h \in H$. Each equivalence class is a discrete subspace of $\mathfrak{U}$ and H operates without fixed points.

DF(5): Given two groups G_1, G_2 and a transformation $\gamma: G_1 \rightarrow G_2$, γ is said to be an *antihomomorphism* if for all $g_i, g_j \in G_1$, $\gamma(g_i g_j) = \gamma(g_j)\gamma(g_i)$. If γ is bijective γ is called an *anti-isomorphism,* and one writes $\gamma: G_1 \overline{\approx}/ G_2$.

TH(7): If $\mathfrak{U}$ is the universal covering space of the locally connected space U, then the fundamental group $\pi(U)$ of U is mapped anti-isomorphically onto H.] Hilton & Wylie (1960), COR(6.7.4).

From RMK(19), TH(7) one deduces that if $\tilde{\mathrm{R}}$ is the space† of all dynamical processes which are physical—in the sense already mentioned—then the fundamental group of $\tilde{\mathrm{R}}$ describes symmetries amongst the intermediate (unobserved) dynamical processes which contribute to the observed, physical dynamical process. We may therefore notice that:

PR(1): The group of physical ordering operators $\breve{\tilde{\mathbf{R}}}$, *which corresponds to the group of physical dynamical processes, is free from all unwanted unobservable ordering operators, and yet its structure is determined by these same unobserved, intermediate, contributory processes.*]

3.6. *Example of a Quotient Theory Language*

In examining the case for an algebraic formulation of quantum field theory, Haag and Kastler (1964) arrive at a condition that involves a quotient structure. They propose to use an abstract C^*-algebra for what is here called the general theory-language £. Their physical theory language is a representation of the abstract C^*-algebra (theory-language), and equivalence in a physical sense is said to hold when two representations of the abstract C^*-algebra have the same kernel. Thus physical equivalence of representations defined above is the analogue of language equivalence in the context of this paper. The kernels of the physical representations must be the same sub-C^*-algebra, and this may be taken as the analogue of the sub-theory-language \$. If \$ $\neq 0$, then one may take the physical structure described by the external quantum numbers as the faithful representation (i.e. at least one such) of £/\$, and the internal quantum numbers will characterise the physical structure represented by \$.

4. *Conclusion*

A formalisation of the heuristic and intuitive ideas about dynamical ordering relations, presented in Part II of this series, has been given in set theoretic language. By the application of the Principle of Corporate Agreement and use of the notion of language equivalence it has been shown that the simple diagrammatic means of describing the ordering of the set

† The operator symbol $\check{}$ has been left off $\check{\mathbf{R}}$, because we do not wish to overtly infer that $\tilde{\mathbf{R}}$ here must be constructed in the same way that $\breve{\tilde{\mathbf{R}}}$ was. Rather the omission may be allowed to infer that the construction of $\breve{\tilde{\mathbf{R}}}$ is a justification for the proposal that we use a quotient structure like $\check{\mathbf{R}}$. Likewise in PR(1), immediately following, the symbol $\check{}$ may be omitted.

of all events detected by all observers is both the simplest and the most informative that can be given.

It has been shown that any causal dynamical theory must have a quotient structure in its theory language, and that this structure is importantly characterised by the fundamental group of the system of dynamical transitions. The fundamental group not only defines the structure of the theory but may also be interpreted as the causal dynamical ordering relation and the associated causal dynamical ordering operator. The Haag-Kastler algebraic formulation of quantum field theory has been seen to provide an example of a quotient theory language.

4.1. *Prospect*

One may hope to come closer to the conventional geometric formalisms of physics (in the differential sense) by exploiting all that is implied of fibre space theory by the introduction of the fundamental group of the system of dynamical transitions. Since a measurement process has been shown in Part I to define a pre-sheaf over the space of physical conditions, one may also exploit the relationships between fibre theory and sheaf theory.

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